

# Perturbation Methods in Stability and Norm Analysis of Spatially Periodic Systems

Makan Fardad and Bassam Bamieh

## Abstract

We consider systems governed by partial differential equations with spatially periodic coefficients over unbounded domains. These spatially periodic systems are considered as perturbations of spatially invariant ones, and we develop perturbation methods to study their stability and  $\mathcal{H}^2$  system norm. The operator Lyapunov equations characterizing the  $\mathcal{H}^2$  norm are studied using a special frequency representation, and formulae are given for the perturbation expansion of their solution. The structure of these equations allows for a recursive method for solving for the expansion terms. Our analysis provides conditions that capture possible resonances between the periodic coefficients and the spatially invariant part of the system. These conditions can be regarded as useful guidelines when spatially periodic coefficients are to be designed to increase/decrease the  $\mathcal{H}^2$  norm of a spatially distributed system. The developed perturbation framework also gives simple conditions for checking exponential stability.

## I. INTRODUCTION

The terms distributed-parameter and infinite-dimensional are used to describe those systems in which the state belongs to an infinite dimensional vector space [1]. Such systems include, but are not limited to, time-delay (retarded) and spatially distributed systems [2]. The latter includes systems in which the dynamics are governed by Partial Differential Equations (PDEs) and it is a subclass of these systems that will be the subject of this study. More specifically, we will analyze certain properties of spatially distributed systems in which the underlying PDEs have spatially periodic coefficients. We refer to such systems as *spatially periodic*. Spatially periodic systems have many real life applications, for example, in boundary layer and channel flow problems with corrugated walls and in nonlinear optics.

Our motivation for this work is to study the effect of spatially periodic coefficients on stability and system norms of spatially distributed systems. This can be thought of as using the periodic coefficients as static feedback controls for spatially distributed systems. For example, in flow control problems where one introduces corrugated wall geometries or spatially periodic body forces, the PDEs that describe the resulting flow dynamics have periodic coefficients that are related to either the wall shapes or the spatially distributed body forces. An important objective is to “design” such wall shapes or body forces to obtain certain stability or instability properties of the resulting dynamics. There are currently no systematic methods for achieving this.

An analogy can be made between the present work and the use of time-periodic coefficients in Ordinary Differential Equations (ODEs). It is known that the introduction of time-periodic coefficients in ODEs with constant coefficients can change the stability properties of the Linear Time Invariant (LTI) system described by the original ODE. A useful picture is to think of an ODE with periodic coefficients as an LTI system modified by time-periodic (memoryless) feedback. It is known that certain unstable LTI systems can be stabilized by being put in feedback with periodic gains of properly designed amplitudes and frequencies [3]. This can be roughly considered as an example of “vibrational control” [4]. On the other hand, certain stable or neutrally stable LTI systems can be destabilized by periodic feedback gains. This is sometimes referred to as “parametric resonance” in the dynamical systems literature [3]. In the above scenarios, the stabilization/destabilization process depends in subtle ways on “resonances” between the natural modes of the LTI subsystem and the frequency and amplitude of the periodic feedback. Although Floquet analysis can be used to ascertain stability of the resulting periodic systems, it is cumbersome to use for *designing* the periodic coefficients. This requires an exhaustive search over frequencies and amplitudes of the periodic coefficients. Alternatively, simple resonance conditions can be

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derived using a perturbation analysis [3], which in turn can be used for a preliminary selection of the coefficient's frequency. In this way, perturbation analysis serves as a useful design tool.

In related recent work [5] we developed computational tools to study stability and system norms for spatially periodic systems. However, for problems where the spatially periodic coefficients are to be designed, using these tools involves a computationally expensive search over spatial frequencies and amplitudes of the coefficients. Therefore, our aim in the present work is to develop a perturbation analysis that can be used to derive resonance conditions and provide a useful design tool in a similar manner to the case of ODEs discussed earlier. These resonance conditions can then identify candidate spatial frequencies to be used for the periodic coefficients. The exact behavior with respect to amplitudes can then be ascertained using the full analysis of [5]. In this manner we reduce the dimension of the search space required for design problems.

Another challenging problem is checking the stability of a spatially periodic, or in general, any infinite-dimensional system. It is well-known that for a finite-dimensional LTI system, the spectrum of the infinitesimal generator (i.e., the  $A$ -matrix) being contained in the open left half of the complex plane is equivalent to exponential stability. In this sense the spectrum of the infinitesimal generator determines stability. Therefore it is said that the system satisfies the *Spectrum Determined Growth Condition* (SDGC). But the SDGC may not hold for some infinite-dimensional LTI systems; indeed the evolution can grow exponentially even though the infinitesimal generator (the  $A$ -operator) has spectrum inside the left half of the complex plane [6]–[8]. In the present work we use perturbation analysis to find simple conditions under which the spatially periodic system satisfies the SDGC and is exponentially stable.

Our presentation is organized as follows. Section II outlines the main results of the paper. Section III briefly reviews the frequency representation of spatially periodic operators. Section IV introduces the problem setup. Section V discusses the analytic perturbation of the  $\mathcal{H}^2$ -norm and Section VI provides related illustrative examples. Section VII studies conditions under which a spatially periodic system is exponentially stable. Many proofs and technical details have been relegated to the Appendix to improve readability.

### Notation

We use  $k \in \mathbb{R}$  to characterize the spatial-frequency variable, also known as the *wave-number*.  $\Sigma(T)$  is the spectrum of the operator  $T$ , and  $\Sigma_p(T)$  its point spectrum,  $\rho(T)$  its resolvent set, and  $R(\zeta, T)$  its resolvent operator  $(\zeta - T)^{-1}$ .  $\mathbb{C}^-$  denotes all complex numbers with real part less than zero, and  $j := \sqrt{-1}$ .  $*$  denotes the complex-conjugate transpose, and also the adjoint of a linear operator.  $\bar{\mathfrak{S}}$  is the closure of the set  $\mathfrak{S} \subset \mathbb{C}$ . We will use the same notation  $A$  for an operator and its kernel representation. The spatio-temporal function  $u(t, x)$  (operator  $A$ ) is denoted by  $\hat{u}(t, k)$  (respectively  $\hat{A}$ ) after the application of a Fourier transform on the spatial variable  $x$ . Where there is no chance of confusion, we use the same notation for a spatially invariant operator and its Fourier symbol.

### Terminology

Throughout the paper, we use the terms *spatial 'operators'* and *spatial 'systems'*. By the former, we mean a *purely spatial* system with no temporal dynamics (i.e. a memoryless operator that acts on a spatial function and yields a spatial function), whereas the latter refers to a *spatio-temporal* system (a system with an internal state evolves on some spatial domain, i.e., for every time  $t$  the state is a function on a spatial domain). When spatially periodic feedback operators are small in norm, we will use the phrases *periodic feedback* and *periodic perturbation* interchangeably. We use the term *pure point spectrum* (or equivalently *discrete spectrum*) to mean that the spectrum of an operator consists entirely of *isolated* eigenvalues [9]. By a *scalar* system, we mean that the Euclidean dimension of the state is equal to one.

## II. MAIN RESULTS

We consider systems described by linear, time-invariant, integro partial differential equations defined on an unbounded one dimensional domain. We use a standard state-space representation of the form

$$\begin{aligned} [\partial_t \psi](t, x) &= [A \psi](t, x) + [B u](t, x), \\ y(t, x) &= [C \psi](t, x), \end{aligned} \tag{1}$$

where  $t \in [0, \infty)$  and  $x \in \mathbb{R}$ ,  $\psi$ ,  $u$ ,  $y$  are spatio-temporal functions, and  $A$ ,  $B$ ,  $C$  are spatial integro-differential operators with coefficients that are periodic functions with a common period  $X$ . We refer to such systems as *spatially periodic*.

It is often physically meaningful to regard the spatially periodic operators as additive or multiplicative perturbations of spatially invariant ones [and by spatially invariant we mean integro-differential operators with constant coefficients]. For example, the generator in (1) can often be decomposed as  $A = A^o + \epsilon E$ , where  $A^o$  is a spatially invariant operator and  $E$  is an operator that includes multiplication by periodic functions. In some control applications, the operator  $E$  is something to be “designed”. Therefore it is desirable to have easily verifiable conditions for stability and norms of such systems. This would then allow for the selection of the spatial period and amplitude of  $E$  to achieve the desired behavior. The perturbation analysis we present, though limited to small values of  $\epsilon$ , provides useful results for selecting candidate “periods” for  $E$ .

Our analysis and results are derived using a special frequency representation. We show that the spatial periodicity of the operators  $A$ ,  $B$  and  $C$  implies that (1) can be rewritten as

$$\begin{aligned} [\partial_t \psi_\theta](t) &= [\mathcal{A}_\theta \psi_\theta](t) + [\mathcal{B}_\theta u_\theta](t), \\ y_\theta(t) &= [\mathcal{C}_\theta \psi_\theta](t), \end{aligned} \quad (2)$$

where  $\theta \in [0, 2\pi/X)$ ; for every value of  $\theta$ ,  $\psi_\theta$ ,  $u_\theta$ ,  $y_\theta$  are bi-infinite vectors, and  $\mathcal{A}_\theta$ ,  $\mathcal{B}_\theta$ ,  $\mathcal{C}_\theta$  are bi-infinite matrices. The systems (2) and (1) are related through a unitary transformation, and in particular quadratic forms and norms are preserved by this transformation. Consequently, stability and quadratic norm properties of (2) and (1) are equivalent. With this transformation, the analysis of the original system (1) is reduced to that of the family of systems (2) that are *decoupled* in the parameter  $\theta$ . In particular, perturbation analysis for (2) is easier and less technical than that for the original system (1).

To make for easier reading we first present the results on perturbation analysis of the  $\mathcal{H}^2$ -norm, and then deal with the issue of stability.

For a large class of infinite-dimensional systems, computing the  $\mathcal{H}^2$ -norm involves solving an *operator* algebraic Lyapunov equation

$$AP + PA^* = -BB^*.$$

In general this is a difficult task that must be done using appropriate discretization techniques. However, in the case when  $A$  and  $B$  are spatially periodic operators, then so is the solution  $P$ . Thus the frequency representation implies that this operator Lyapunov equation is equivalent to the decoupled family of matrix Lyapunov equations

$$\mathcal{A}_\theta \mathcal{P}_\theta + \mathcal{P}_\theta \mathcal{A}_\theta^* = -\mathcal{B}_\theta \mathcal{B}_\theta^*, \quad (3)$$

where  $\mathcal{A}_\theta$ ,  $\mathcal{B}_\theta$  and  $\mathcal{P}_\theta$  are the bi-infinite matrix representations of  $A$ ,  $B$  and  $P$ . Once  $\mathcal{P}_\theta$  is found, the  $\mathcal{H}^2$ -norm of the system can be computed from [5]

$$\frac{1}{2\pi} \int_0^\Omega \text{trace}[\mathcal{C}_\theta \mathcal{P}_\theta \mathcal{C}_\theta^*] d\theta, \quad \Omega = 2\pi/X.$$

Solving (3) is still a difficult problem in general since it involves bi-infinite matrices. We use perturbation analysis as follows: the generator is expressed as  $\mathcal{A}_\theta = \mathcal{A}_\theta^o + \epsilon \mathcal{E}_\theta$  where  $\mathcal{A}_\theta^o$  and  $\mathcal{E}_\theta$  correspond to the spatially invariant and spatially periodic components respectively. It follows that the solution  $\mathcal{P}_\theta$  is analytic in  $\epsilon$  and the terms of its power series expansion (denoted by  $\mathcal{P}_\theta^{(i)}$ ) satisfy a sequence of forward coupled Lyapunov equations. Furthermore, the terms  $\mathcal{P}_\theta^{(i)}$  are banded matrices, with the number of bands increasing with the index  $i$ . These Lyapunov equations can then be solved recursively for  $i = 0, 1, 2, \dots$ . Formulae for these representations and the corresponding sequence of Lyapunov equations are given in Section V. In some examples that we present in Section VI, these formulae lead to simple “resonance” conditions for stabilization or destabilization of PDEs using spatially periodic feedback.

The second set of results concern the problem of stability. As mentioned in the Introduction, when  $A$  is an infinite-dimensional operator it is possible that its spectrum  $\Sigma(A)$  lies inside  $\mathbb{C}^-$ , and yet  $\|e^{At}\|$  grows exponentially [6]–[8]. In such cases it is said that the *spectrum-determined growth condition* is not satisfied [8]. Yet there exists a wide range of infinite-dimensional systems for which the spectrum-determined growth condition

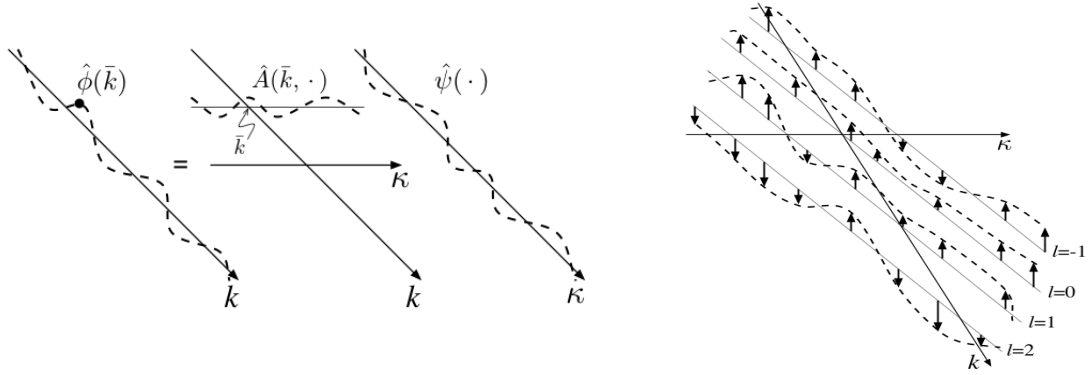


Fig. 1. Left: Pictorial representation of  $\hat{\phi}(\bar{k}) = \int_{\mathbb{R}} \hat{A}(\bar{k}, \kappa) \hat{\psi}(\kappa) d\kappa$ . Right: The frequency kernel  $\hat{A}$  of a spatially periodic operator  $A$ .

is satisfied. These include (but are not limited to) systems for which the  $A$ -operator is *sectorial* (also known as an operator which generates a *holomorphic* or *analytic* semigroup) [9]–[11] or is a *Reisz-spectral* operator [2]. In this paper we focus on sectorial operators.

Thus to establish exponential stability of a system, one possibility would be to show simultaneously that

- (i)  $A$  is sectorial,
- (ii)  $\Sigma(A)$  lies in  $\mathbb{C}^-$ .

But this still does not make the problem trivial. In fact proving that an infinite-dimensional operator is sectorial, and then finding its spectrum, can in general be extremely difficult.

Once again we use perturbation methods to show (i) and (ii). We consider  $A$  to have the form  $A = A^0 + \epsilon E$  where  $A^0$  is a spatially invariant operator,  $E$  is a spatially periodic operator, and  $\epsilon$  is a small complex scalar. Using the bi-infinite matrix representation, we first find conditions on the spatially invariant operator  $A^0$  such that (i) and (ii) are satisfied. We then show that (i) and (ii) will *remain* satisfied if  $\epsilon$  is small enough and if the spatially periodic operator  $E$  is ‘weaker’ than  $A^0$  (in the sense that  $E$  is relatively bounded with respect to  $A^0$ ). The utility of this approach is that (i) and (ii) are much easier to check for a spatially invariant operator than they are for a spatially periodic one.

### III. FREQUENCY REPRESENTATION OF PERIODIC OPERATORS

In this section we briefly discuss the frequency domain representation of spatially periodic operators. We then show how this representation can be used to convert a spatially periodic system into a family of matrix-valued LTI systems. For a detailed account the reader is referred to [5] and [12].

Let  $\hat{\psi}(k)$  and  $\hat{\phi}(k)$  denote the Fourier transforms of two spatial functions  $\psi(x)$  and  $\phi(x)$  respectively. If  $\psi$  and  $\phi$  are related by a linear operator,  $\phi = A\psi$ , then so are  $\hat{\psi}$  and  $\hat{\phi}$  and we have

$$\phi(x) = \int_{\mathbb{R}} A(x, \chi) \psi(\chi) d\chi \xleftrightarrow{\mathcal{F}_x} \hat{\phi}(k) = \int_{\mathbb{R}} \hat{A}(k, \kappa) \hat{\psi}(\kappa) d\kappa \quad (4)$$

where  $A$  and  $\hat{A}$  are *kernel functions* in the spatial and Fourier domains, respectively; see Figure 1 (left). It is shown in [5] [12] that the most general spatially periodic operator  $A$  with spatial period  $X = 2\pi/\Omega$  can be represented in the Fourier domain as an operator with a kernel function of the form

$$\hat{A}(k, \kappa) = \sum_{l \in \mathbb{Z}} \hat{A}_l(k) \delta(k - \kappa - \Omega l). \quad (5)$$

Thus the kernel function corresponding to  $\hat{A}$  is composed of parallel and equally-spaced ‘impulse sheets’ which can be visualized in Figure 1 (right). [5] further describes how the particular structure (5) of  $\hat{A}$  allows (4) to be

given a matrix representation

$$\begin{bmatrix} \vdots \\ \hat{\phi}(\theta - \Omega) \\ \hat{\phi}(\theta) \\ \hat{\phi}(\theta + \Omega) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & \ddots \\ & \hat{A}_0(\theta - \Omega) & \hat{A}_{-1}(\theta - \Omega) & \hat{A}_{-2}(\theta - \Omega) & \\ & & \hat{A}_1(\theta) & \hat{A}_0(\theta) & \hat{A}_{-1}(\theta) \\ & & & \hat{A}_2(\theta + \Omega) & \hat{A}_1(\theta + \Omega) & \hat{A}_0(\theta + \Omega) \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{\psi}(\theta - \Omega) \\ \hat{\psi}(\theta) \\ \hat{\psi}(\theta + \Omega) \\ \vdots \end{bmatrix}, \quad \theta \in [0, \Omega), \quad (6)$$

for which we adopt the notation

$$\phi_\theta = \mathcal{A}_\theta \psi_\theta.$$

In other words, a general spatially periodic operator  $A$  can be described by a family of (bi-infinite) matrices  $\mathcal{A}_\theta$  parameterized by a variable  $\theta$ .

*Remark 1:* We emphasize that the  $\hat{A}_l(\cdot)$  in (5) and (6) can be matrices. Thus, in general,  $\mathcal{A}_\theta$  has a “block” structure. But throughout this paper and for the sake of simplicity, we choose not to explicitly refer to this block structure, even though all our results hold for matrix-valued  $\hat{A}_l(\cdot)$ . In the same light, we do not refer directly to the Euclidean dimension of the vectors  $\hat{\psi}(\cdot)$  and  $\hat{\phi}(\cdot)$ . ■

*Spatially invariant* [13] and *spatially periodic pure multiplication* operators constitute special subclasses of spatially periodic operators. In the framework established above,  $\mathcal{A}_\theta$  is *diagonal* for spatially invariant operators, and *Toeplitz* for periodic pure multiplication operators.

*Example 1:*  $A = \partial_x$  and  $F(x) = \cos(\Omega x)$  have the following representations

$$\mathcal{A}_\theta = \begin{bmatrix} \ddots & & & \\ & j\theta + j\Omega n & & \\ & & \ddots & \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} \ddots & & & \\ & 0 & \frac{1}{2} & \\ & \frac{1}{2} & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

for every  $\theta \in [0, \Omega)$ , respectively. Notice that since  $A$  is spatially invariant, it is fully described by its *Fourier symbol*  $\hat{A}_0(k) = jk$ ,  $k \in \mathbb{R}$ . And it is the samples of  $\hat{A}_0(\cdot)$  that make up the diagonal of  $\mathcal{A}_\theta$  for every  $\theta$ . We have dropped the  $\theta$  subscript in  $\mathcal{F}$ , as it is independent of this variable. ■

*Remark 2:* It is possible to define a unitary operator  $\mathcal{M}_\theta$  [5] such that  $\psi_\theta = \mathcal{M}_\theta \hat{\psi}$ ,  $\phi_\theta = \mathcal{M}_\theta \hat{\phi}$ , and thus  $\mathcal{A}_\theta = \mathcal{M}_\theta \hat{A} \mathcal{M}_\theta^*$ .  $\mathcal{M}_\theta$  is equivalent to the *frequency domain lifting* operation of [14] and [15] (see also [16]). By the unitary property of the lifting operator it follows that

$$\sum_{l \in \mathbb{Z}} \int_{\mathbb{R}} \text{trace}[\hat{A}_l(k) \hat{A}_l^*(k)] dk = \int_0^\Omega \text{trace}[\mathcal{A}_\theta \mathcal{A}_\theta^*] d\theta = \int_0^\Omega \|\mathcal{A}_\theta\|_{\text{HS}}^2 d\theta, \quad (7)$$

with  $\|T\|_{\text{HS}}^2 := \text{trace}[T T^*]$  being the square of the Hilbert-Schmidt norm<sup>1</sup> of  $T$ . ■

Finally, given a spatially periodic system in state-space form (1) with spatially periodic  $A$ ,  $B$ , and  $C$ , one can replace each of these operators with its bi-infinite matrix representation to obtain the family LTI systems (2).

*Example 2:* Consider the spatially periodic heat equation on  $x \in \mathbb{R}$

$$\begin{aligned} \partial_t \psi(t, x) &= (\partial_x^2 - \alpha \cos(\Omega x)) \psi(t, x) + u(t, x), \\ y(t, x) &= \psi(t, x), \end{aligned} \quad (8)$$

with real  $\alpha \neq 0$  and  $\Omega > 0$ .<sup>2</sup> Clearly  $A = \partial_x^2 + \alpha \cos(\Omega x)$  with domain

$$\mathcal{D} = \{\phi \in L^2(\mathbb{R}) \mid \phi, \frac{d\phi}{dx} \text{ absolutely continuous, } \frac{d^2\phi}{dx^2} \in L^2(\mathbb{R})\},$$

<sup>1</sup>The Hilbert-Schmidt norm of an operator is a generalization of the Frobenius norm of finite-dimensional matrices  $\|A\|_{\text{F}}^2 = \sum_{m,n} |a_{mn}|^2 = \text{trace}[A A^*]$ .

<sup>2</sup>By  $\partial_t \psi(t, x)$  and  $\partial_x^2 \psi(t, x)$  we mean the spatio-temporal functions  $\partial_t \psi$  and  $\partial_x^2 \psi$  evaluated at the point  $(t, x)$ .

$B = C = \delta(x)$  are the identity convolution operator on  $L^2(\mathbb{R})$ . Rewriting the system in its matrix representation

$$\begin{aligned}\partial_t \psi_\theta(t) &= \mathcal{A}_\theta \psi_\theta(t) + \mathcal{B}_\theta u_\theta(t), \\ y_\theta(t) &= \mathcal{C}_\theta \psi_\theta(t),\end{aligned}\tag{9}$$

with

$$\mathcal{A}_\theta = \begin{bmatrix} \ddots & & \\ & -(\theta + \Omega n)^2 & \\ & & \ddots \end{bmatrix} - \alpha \begin{bmatrix} \ddots & \ddots & & \\ & 0 & \frac{1}{2} & \\ & \frac{1}{2} & 0 & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \quad \mathcal{B}_\theta = \mathcal{C}_\theta = \begin{bmatrix} \ddots & & \\ & 1 & \\ & & \ddots \end{bmatrix},\tag{10}$$

$$G_\theta(\omega) := \mathcal{C}_\theta (j\omega \mathcal{I} - \mathcal{A}_\theta)^{-1} \mathcal{B}_\theta = \begin{bmatrix} \ddots & \ddots & & \\ & \ddots & \ddots & \\ & & \frac{\alpha}{2} & \\ \frac{\alpha}{2} & j\omega + (\theta + \Omega n)^2 & \frac{\alpha}{2} & \\ & \frac{\alpha}{2} & \ddots & \ddots \end{bmatrix}^{-1}.\tag{11}$$

Notice that (9)–(10) is now fully decoupled in the variable  $\theta$ . In other words, (8) is equivalent to the family of state-space representations (9)–(10), and transfer function (11), all parameterized by the variable  $\theta \in [0, \Omega)$ . ■

#### IV. PROBLEM SETUP

Let us now consider a system of the form

$$\begin{aligned}\partial_t \psi(t, x) &= A \psi(t, x) + B u(t, x) \\ &= (A^\circ + B^\circ \epsilon F C^\circ) \psi(t, x) + B u(t, x), \\ y(t, x) &= C \psi(t, x),\end{aligned}\tag{12}$$

where  $t \in [0, \infty)$  and  $x \in \mathbb{R}$  with the following assumptions. The (possibly unbounded) operators  $A^\circ, B^\circ, C^\circ$  are spatially invariant, and the bounded operators  $B, C$  are spatially periodic.  $F(x) = 2L \cos(\Omega x)$  with  $L$  a constant matrix, and  $\epsilon$  is a complex scalar.  $A^\circ, B^\circ, C^\circ$  and  $E := B^\circ F C^\circ$  are all defined on a dense domain  $\mathcal{D} \subset L^2(\mathbb{R})$  and are closed.  $u, y$ , and  $\psi$  are the spatio-temporal input, output, and state of the system, respectively. We will refer to  $A$  as the *infinitesimal generator* of the system.

*Comment on Notation:* To avoid clutter, we henceforth drop the “ $\hat{\cdot}$ ” on the Fourier symbol of operators and frequency domain functions. For example, we use  $A_0(\cdot)$  [instead of  $\hat{A}_0(\cdot)$ ] to represent the Fourier symbol of the spatially invariant operator  $A^\circ$ . ■

As shown in detail in [5] and also briefly in the previous section, the system can be represented in the (spatial) Fourier domain by the family of systems

$$\begin{aligned}\partial_t \psi_\theta(t) &= \mathcal{A}_\theta \psi_\theta(t) + \mathcal{B}_\theta u_\theta(t) \\ &= (\mathcal{A}_\theta^\circ + \epsilon \mathcal{B}_\theta^\circ \mathcal{F} \mathcal{C}_\theta^\circ) \psi_\theta(t) + \mathcal{B}_\theta u_\theta(t) \\ y_\theta(t) &= \mathcal{C}_\theta \psi_\theta(t),\end{aligned}\tag{13}$$

parameterized by  $\theta \in [0, \Omega)$ . Here  $\mathcal{B}_\theta$  and  $\mathcal{C}_\theta$  have the general form of the operator in (6) [i.e. can possess any

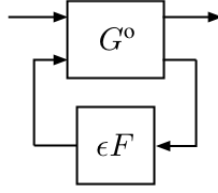


Fig. 2.  $G^o$  has a spatially invariant infinitesimal generator  $A^o$ . The LFT of  $G^o$  and the spatially periodic multiplication operator  $\epsilon F$  yields a system which has a spatially periodic infinitesimal generator  $A$ .

number of nonzero sbudiagonals],  $\mathcal{F}$  has the form given in Example 1 with  $\frac{1}{2}$  replaced by  $L$ , and

$$\mathcal{A}_\theta^o = \begin{bmatrix} \ddots & & \\ & A_0(\theta_n) & \\ & & \ddots \end{bmatrix}, \quad \mathcal{B}_\theta^o = \begin{bmatrix} \ddots & & \\ & B^o(\theta_n) & \\ & & \ddots \end{bmatrix}, \quad \mathcal{C}_\theta^o = \begin{bmatrix} \ddots & & \\ & C^o(\theta_n) & \\ & & \ddots \end{bmatrix}, \quad \mathcal{E}_\theta := \mathcal{B}_\theta^o \mathcal{F} \mathcal{C}_\theta^o = \begin{bmatrix} \ddots & & & \\ & 0 & A_{-1}(\theta_n) & \\ & & 0 & \ddots \\ & A_1(\theta_n) & 0 & \\ & & & \ddots & \ddots \end{bmatrix} \quad (14)$$

with  $\theta_n := \theta + \Omega n$ ,  $n \in \mathbb{Z}$ , and

$$A_1(\cdot) := B^o(\cdot) L C^o(\cdot - \Omega), \quad A_{-1}(\cdot) := B^o(\cdot) L C^o(\cdot + \Omega). \quad (15)$$

We emphasize that the convention used in the representation of  $\mathcal{E}_\theta$  in (14) is the same as that used in (6); for example the  $n^{\text{th}}$  row of  $\mathcal{E}_\theta$  is  $\{\dots, 0, A_1(\theta_n), 0, A_{-1}(\theta_n), 0, \dots\}$ .

*Remark 3:* We note that taking  $F(x)$  to be a pure cosine is not restrictive. The results obtained here can be easily extended to problems where  $F(x)$  is any periodic function with absolutely convergent Fourier series coefficients. The inclusion of higher harmonics of  $\Omega$  in  $F(x)$ , namely functions of frequency  $2\Omega$ ,  $3\Omega$ , etc, would not reveal new interesting phenomena and would only complicate the algebra. ■

*Remark 4:* The system (12) can be considered as the LFT (linear fractional transformation [17]) of a spatially periodic system  $G^o$  with *spatially invariant* infinitesimal generator  $A^o$ ,

$$G^o = \left[ \begin{array}{c|cc} A^o & B & B^o \\ \hline C & 0 & 0 \\ C^o & 0 & 0 \end{array} \right]$$

and the (memoryless and bounded) spatially periodic pure multiplication operator  $\epsilon F(x) = \epsilon 2L \cos(\Omega x)$ , see Figure 2. ■

### Stability Analysis and Sectorial Operators

It is shown in [5] that for a general spatially periodic operator  $A$  we have

$$\Sigma(A) = \overline{\bigcup_{\theta \in [0, \Omega)} \Sigma(\mathcal{A}_\theta)}. \quad (16)$$

In the case where  $A$  is spatially invariant [and thus  $\mathcal{A}_\theta = \text{diag}\{\dots, A_0(\theta_n), \dots\}$ ], (16) further simplifies to

$$\Sigma(A) = \overline{\bigcup_{k \in \mathbb{R}} \Sigma_p(A_0(k))}. \quad (17)$$

*Example 3:* Let  $A = -(\partial_x^2 + \varkappa^2)^2$ . Then  $A_0(k) = -(k^2 - \varkappa^2)^2$ , see Figure 3 (left). On the other hand, since  $A_0(\cdot)$  is scalar,  $\Sigma_p(A_0(k)) = A_0(k)$  for every  $k$ . It is easy to see that  $A_0(\cdot)$  takes every real negative value and thus from (17)  $A$  has the *continuous spectrum*  $\Sigma(A) = (-\infty, 0]$ , see Figure 3 (center). ■

*Remark 5:* When  $A$  is *spatially invariant*, a helpful way to think about  $\Sigma(A)$  in terms of its symbol  $A_0$  is suggested by the previous example. First plot  $\Sigma_p(A_0(\cdot))$  in the ‘complex-plane  $\times$  spatial-frequency’ space, as in



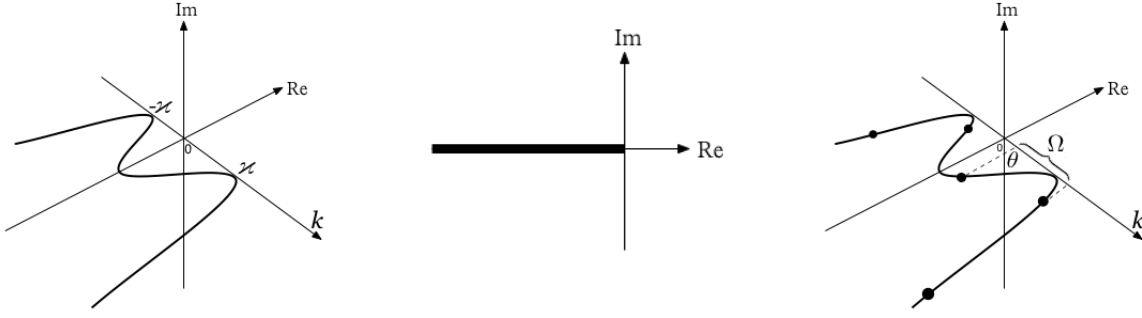


Fig. 3. Left: Representation of the symbol  $A_0(\cdot)$  of Example 3 in ‘complex-plane  $\times$  spatial-frequency’ space. Center:  $\Sigma(A)$  in the complex plane. Right: For spatially invariant  $A$ , the (diagonal) elements of  $\mathcal{A}_\theta$  are samples of the Fourier symbol  $A_0(\cdot)$ .

Figure 3 (left) of Example 3. Then the orthogonal projection onto the complex plane of this plot would give  $\Sigma(A)$ , as in Figure 3 (center). This can be considered as a geometric interpretation of (17). In Example 3, since  $A_0(\cdot)$  is real scalar and takes only negative values, this projection yields only the negative real axis. But in general if  $A_0(\cdot) \in \mathbb{C}^{q \times q}$ , this projection would lead to  $q$  curves in the complex plane. Notice also that in this setting,  $\Sigma(\mathcal{A}_\theta)$  is the projection onto the complex plane of samples of  $\Sigma_p(A_0(\cdot))$  taken at  $k = \theta_n = \theta + \Omega n$ ,  $n \in \mathbb{Z}$ , in the ‘complex-plane  $\times$  spatial-frequency’ space. This can be considered as a geometric interpretation of (16). Figure 3 (right) shows these samples in the ‘complex-plane  $\times$  spatial-frequency’ space for a scalar  $A$ . ■

We next introduce a special subclass of *holomorphic* (or *analytic*) semigroups. The reader is referred to [9]–[11] for a detailed discussion. Suppose  $A$  is densely defined,  $\rho(A)$  contains a sector of the complex plane  $|\arg(z - \alpha)| \leq \frac{\pi}{2} + \gamma$ ,  $\gamma > 0$ ,  $\alpha \in \mathbb{R}$ , and there exists some  $M > 0$  such that

$$\|(zI - A)^{-1}\| \leq \frac{M}{|z - \alpha|} \quad \text{for } |\arg(z - \alpha)| \leq \frac{\pi}{2} + \gamma. \quad (18)$$

Then  $A$  generates a holomorphic semigroup and we write  $A \in \mathcal{H}(\gamma, \alpha, M)$  [11] [9]. We say that  $A$  is *sectorial* if  $A$  belongs to some  $\mathcal{H}(\gamma, \alpha, M)$ .

Finally, a semigroup is called exponentially stable if there exist positive constants  $M$  and  $\varrho$  such that [2]

$$\|e^{At}\| \leq M e^{-\varrho t} \quad \text{for } t \geq 0.$$

**Theorem 1:** Let  $A$  be sectorial. Then if  $\Sigma(A) \subset \mathbb{C}^-$ ,  $A$  generates an exponentially stable semigroup.

*Proof:* If  $A$  is sectorial it defines a holomorphic semigroup and thus  $e^{At}$  is differentiable for  $t > 0$  [10]. Then [8] shows that this is sufficient for the spectrum-determined growth condition to hold. In particular, if  $\Sigma(A) \subset \mathbb{C}^-$ ,  $A$  generates an exponentially stable semigroup. ■

### $\mathcal{H}^2$ -Norm of Spatially Periodic Systems

We define the  $\mathcal{H}^2$ -norm of an exponentially stable spatially periodic system  $G$  as

$$\|G\|_{\mathcal{H}^2}^2 = \frac{1}{2\pi} \int_0^\Omega \int_0^\infty \text{trace}[\mathcal{G}_\theta(t) \mathcal{G}_\theta^*(t)] dt d\theta,$$

where  $\mathcal{G}_\theta(t) = \mathcal{C}_\theta e^{A_\theta t} \mathcal{B}_\theta$  is the impulse response of the system (13). The intuition for this definition and the proof of the following theorem can be found in [5].

**Theorem 2:** Consider the exponentially stable spatially periodic LTI system  $G$ , with spatial period  $X = 2\pi/\Omega$  and state-space realization (12)–(13). We have

$$\|G\|_{\mathcal{H}^2}^2 = \frac{1}{2\pi} \int_0^\Omega \text{trace}[\mathcal{C}_\theta \mathcal{P}_\theta \mathcal{C}_\theta^*] d\theta = \frac{1}{2\pi} \int_0^\Omega \text{trace}[\mathcal{B}_\theta^* \mathcal{Q}_\theta \mathcal{B}_\theta] d\theta,$$

where  $\mathcal{P}_\theta$  and  $\mathcal{Q}_\theta$  are the solutions of the ( $\theta$ -parameterized) algebraic Lyapunov equations

$$\mathcal{A}_\theta \mathcal{P}_\theta + \mathcal{P}_\theta \mathcal{A}_\theta^* = -\mathcal{B}_\theta \mathcal{B}_\theta^*, \quad \mathcal{A}_\theta^* \mathcal{Q}_\theta + \mathcal{Q}_\theta \mathcal{A}_\theta = -\mathcal{C}_\theta^* \mathcal{C}_\theta.$$

## V. PERTURBATION ANALYSIS OF THE $\mathcal{H}^2$ -NORM

The difficulty in calculating the  $\mathcal{H}^2$ -norm using Theorem 2, is that unless  $\mathcal{A}_\theta$ ,  $\mathcal{B}_\theta$  and  $\mathcal{C}_\theta$  are diagonal (i.e.,  $G$  is a spatially invariant system), the operators  $\mathcal{P}_\theta$  and  $\mathcal{Q}_\theta$  are “full”, meaning that they possess *all* of their (infinite number of) subdiagonals. This makes the computation of the  $\mathcal{H}^2$ -norm numerically intensive. Namely, one has to solve an *infinite-dimensional* algebraic Lyapunov equation to find the operator  $\mathcal{P}_\theta$  (or  $\mathcal{Q}_\theta$ ) *for every value of*  $\theta \in [0, \Omega]$ . In this section we will see how one can use analytic perturbation techniques to compute the  $\mathcal{H}^2$ -norm in a simple and numerically efficient way, and without having to explicitly find the full  $\mathcal{P}_\theta$  and  $\mathcal{Q}_\theta$  operators. Such a perturbation analysis is very useful in predicting general trends and extracting valuable information about the  $\mathcal{H}^2$ -norm.

Let us now consider the general setup of (12), where we take  $\epsilon$  is a small *real* scalar. We also assume that both  $A^0$  and  $A = A^0 + \epsilon E$  define exponentially stable strongly continuous semigroups (also known as  $C_0$ -semigroups) on  $L^2(\mathbb{R})$  [2], and that  $B$  and  $C$  are spatially invariant operators. We are interested in the changes in the  $\mathcal{H}^2$ -norm of this system for small magnitudes of  $\epsilon$  and different values of the frequency  $\Omega$ .

Let us define

$$\mathcal{P}_\theta(\epsilon) := \mathcal{P}_\theta^{(0)} + \epsilon \mathcal{P}_\theta^{(1)} + \epsilon^2 \mathcal{P}_\theta^{(2)} + \dots,$$

with  $\mathcal{P}_\theta^*(\epsilon) = \mathcal{P}_\theta(\epsilon)$ . Notice that this implies  $\mathcal{P}_\theta^{(m)*} = \mathcal{P}_\theta^{(m)}$  for all  $m = 0, 1, 2, \dots$ . Our aim is to find  $\mathcal{P}_\theta^{(m)}$  by solving the Lyapunov equation

$$\mathcal{A}_\theta(\epsilon) \mathcal{P}_\theta(\epsilon) + \mathcal{P}_\theta(\epsilon) \mathcal{A}_\theta^*(\epsilon) \equiv -\mathcal{B}_\theta \mathcal{B}_\theta^* \quad (19)$$

$\Downarrow$

$$\begin{aligned} & (\mathcal{A}_\theta^0 + \epsilon \mathcal{E}_\theta) (\mathcal{P}_\theta^{(0)} + \epsilon \mathcal{P}_\theta^{(1)} + \epsilon^2 \mathcal{P}_\theta^{(2)} + \dots) + \\ & (\mathcal{P}_\theta^{(0)} + \epsilon \mathcal{P}_\theta^{(1)} + \epsilon^2 \mathcal{P}_\theta^{(2)} + \dots) (\mathcal{A}_\theta^0 + \epsilon \mathcal{E}_\theta)^* \equiv -\mathcal{B}_\theta \mathcal{B}_\theta^*, \end{aligned} \quad (20)$$

and compute the  $\mathcal{H}^2$ -norm of the system using Theorem 2

$$\|G\|_{\mathcal{H}^2}^2 = \frac{1}{2\pi} \int_0^\Omega \text{trace}[\mathcal{C}_\theta \mathcal{P}_\theta(\epsilon) \mathcal{C}_\theta^*] d\theta.$$

It is easy to see from (20) that

$$\mathcal{A}_\theta^0 \mathcal{P}_\theta^{(0)} + \mathcal{P}_\theta^{(0)} \mathcal{A}_\theta^{0*} = -\mathcal{B}_\theta \mathcal{B}_\theta^*, \quad (21)$$

$$\mathcal{A}_\theta^0 \mathcal{P}_\theta^{(1)} + \mathcal{P}_\theta^{(1)} \mathcal{A}_\theta^{0*} = -(\mathcal{E}_\theta \mathcal{P}_\theta^{(0)} + \mathcal{P}_\theta^{(0)} \mathcal{E}_\theta^*), \quad (22)$$

$$\mathcal{A}_\theta^0 \mathcal{P}_\theta^{(2)} + \mathcal{P}_\theta^{(2)} \mathcal{A}_\theta^{0*} = -(\mathcal{E}_\theta \mathcal{P}_\theta^{(1)} + \mathcal{P}_\theta^{(1)} \mathcal{E}_\theta^*), \quad (23)$$

$\vdots$

Now since  $\mathcal{A}_\theta^0$  and  $\mathcal{B}_\theta \mathcal{B}_\theta^*$  are diagonal in (21), so is  $\mathcal{P}_\theta^{(0)}$ . In (22), the right hand side operator has the structure of being nonzero only on the first upper and lower subdiagonals, and hence  $\mathcal{P}_\theta^{(1)}$  inherits the same structure (since  $\mathcal{A}_\theta^0$  is diagonal). In the same manner, one can show that  $\mathcal{P}_\theta^{(2)}$  is only nonzero on the main diagonal and the second upper and lower subdiagonals, and so on for other  $\mathcal{P}_\theta^{(m)}$ . We have

$$\mathcal{P}_\theta^{(0)} = \begin{bmatrix} \ddots & & & \\ & \ddots & & \\ & & P_0(\theta_n) & \\ & & & \ddots \end{bmatrix}, \quad \mathcal{P}_\theta^{(1)} = \begin{bmatrix} \ddots & \ddots & & \\ & \ddots & & \\ & & 0 & P_1^*(\theta_n) \\ & & P_1(\theta_n) & 0 \\ & & & \ddots & \ddots \end{bmatrix}, \quad \mathcal{P}_\theta^{(2)} = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & 0 & P_2^*(\theta_n) \\ & & \ddots & 0 & Q_0(\theta_n) \\ & & 0 & Q_0(\theta_n) & 0 \\ & & P_2(\theta_n) & 0 & \ddots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \quad \dots$$

where the  $n^{\text{th}}$  row of  $\mathcal{P}_\theta^{(1)}$  is equal to  $\{\cdots, 0, P_1(\theta_n), 0, P_1^*(\theta_n), 0, \cdots\}$ , and the  $n^{\text{th}}$  row of  $\mathcal{P}_\theta^{(2)}$  is equal to  $\{\cdots, 0, P_2(\theta_n), 0, Q_0(\theta_n), 0, P_2^*(\theta_n), 0, \cdots\}$ .

*Remark 6:* It is important to realize that, not only is  $\mathcal{P}_\theta^{(m)}$  not a “full” operator, it has at most  $m$  nonzero upper and lower subdiagonals. Also, all  $\mathcal{P}_\theta^{(m)}$  for odd  $m$  have zero diagonal and are thus trace-free operators. ■

*Remark 7:* Note that even though  $\mathcal{A}_\theta = \mathcal{A}_\theta^0 + \epsilon \mathcal{E}_\theta$  has only one nonzero subdiagonal [see (14)],  $\mathcal{P}_\theta(\epsilon) = \mathcal{P}_\theta^{(0)} + \epsilon \mathcal{P}_\theta^{(1)} + \cdots$  possesses all of its subdiagonals. This is precisely the reason why direct calculation of  $\mathcal{P}_\theta$  in Theorem 2 is computationally difficult. ■

Now returning to (21)-(23),  $\mathcal{P}_\theta^{(0)}$ ,  $\mathcal{P}_\theta^{(1)}$  and  $\mathcal{P}_\theta^{(2)}$  are found by equating, element by element, the bi-infinite matrices on both sides of these equations. For example, (21) leads to

$$A_0(\theta + \Omega n) P_0(\theta + \Omega n) + P_0(\theta + \Omega n) A_0^*(\theta + \Omega n) = -B(\theta + \Omega n) B^*(\theta + \Omega n)$$

for every  $n \in \mathbb{Z}$ , and  $\theta \in [0, \Omega)$ . But notice that as  $n$  assumes values over all integers and  $\theta$  changes in  $[0, \Omega)$ ,  $k = \theta + \Omega n$  takes all real values, and one can rewrite the above equation as

$$A_0(k) P_0(k) + P_0(k) A_0^*(k) = -B(k) B^*(k),$$

for all  $k \in \mathbb{R}$ . Applying the same procedure to (22)-(23), one arrives at

$$A_0(k) P_0(k) + P_0(k) A_0^*(k) = -B(k) B^*(k), \quad (24)$$

$$A_0(k) P_1(k) + P_1(k) A_0^*(k - \Omega) = -\left(A_1(k) P_0(k - \Omega) + P_0(k) A_{-1}^*(k - \Omega)\right), \quad (25)$$

$$A_0(k) Q_0(k) + Q_0(k) A_0^*(k) = -\left(A_1(k) P_1^*(k) + P_1(k) A_1^*(k) + A_{-1}(k) P_1(k + \Omega) + P_1^*(k + \Omega) A_{-1}^*(k)\right), \quad (26)$$

and so on for all nonzero diagonals of  $\mathcal{P}_\theta^{(m)}$ ,  $m = 3, 4, \dots$ .

*Remark 8:* Notice that from the above equations, one first finds  $P_0(\cdot)$  from (24), then  $P_1(\cdot)$  from (25), and so on. In other words, computing the subdiagonals of  $\mathcal{P}_\theta$  becomes “decoupled” in one direction. This decoupling would not have been possible had we not employed a perturbation approach and had attempted to solve (19) directly. ■

Returning to the calculation of the  $\mathcal{H}^2$ -norm, let us first separate the diagonal part of  $\mathcal{P}_\theta^{(2)}$  by rewriting it as  $\mathcal{P}_\theta^{(2)} = \overline{\mathcal{P}}_\theta^{(2)} + \widetilde{\mathcal{P}}_\theta^{(2)}$ , where

$$\overline{\mathcal{P}}_\theta^{(2)} := \begin{bmatrix} \ddots & & \\ & Q_0(\theta_n) & \\ & & \ddots \end{bmatrix},$$

and  $\widetilde{\mathcal{P}}_\theta^{(2)}$  contains the rest of  $\mathcal{P}_\theta^{(2)}$ . Clearly  $\text{trace}[\mathcal{C}_\theta \widetilde{\mathcal{P}}_\theta^{(2)} \mathcal{C}_\theta^*] = 0$ . Also, recall that

$$\text{trace}[\mathcal{C}_\theta \mathcal{P}_\theta^{(2m+1)} \mathcal{C}_\theta^*] = 0, \quad m = 0, 1, 2, \dots \quad (27)$$

Now one can write the following

$$\begin{aligned} \|G\|_{\mathcal{H}^2}^2 &= \frac{1}{2\pi} \int_0^\Omega \text{trace}[\mathcal{C}_\theta \mathcal{P}_\theta(\epsilon) \mathcal{C}_\theta^*] d\theta \\ &= \frac{1}{2\pi} \int_0^\Omega \text{trace}[\mathcal{C}_\theta \mathcal{P}_\theta^{(0)} \mathcal{C}_\theta^* + \epsilon^2 \mathcal{C}_\theta \mathcal{P}_\theta^{(2)} \mathcal{C}_\theta^*] d\theta + O(\epsilon^4) \\ &= \frac{1}{2\pi} \int_0^\Omega \text{trace}[\mathcal{C}_\theta \mathcal{P}_\theta^{(0)} \mathcal{C}_\theta^* + \epsilon^2 \mathcal{C}_\theta \overline{\mathcal{P}}_\theta^{(2)} \mathcal{C}_\theta^*] d\theta + O(\epsilon^4), \end{aligned}$$

where the absence of odd powers of  $\epsilon$  is due to (27), and the last equation follows from the fact that  $\text{trace}[\mathcal{C}_\theta \widetilde{\mathcal{P}}_\theta^{(2)} \mathcal{C}_\theta^*] = 0$ . Next, using the unitaryness of the lifting transform we have

$$\begin{aligned} \int_0^\Omega \text{trace}[\mathcal{C}_\theta \mathcal{P}_\theta^{(0)} \mathcal{C}_\theta^*] d\theta &= \int_{-\infty}^\infty \text{trace}[C(k) P_0(k) C^*(k)] dk, \\ \int_0^\Omega \text{trace}[\mathcal{C}_\theta \overline{\mathcal{P}}_\theta^{(2)} \mathcal{C}_\theta^*] d\theta &= \int_{-\infty}^\infty \text{trace}[C(k) Q_0(k) C^*(k)] dk, \end{aligned}$$

and we arrive at the final result

$$\|G\|_{\mathcal{H}^2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[C(k)P_0(k)C^*(k) + \epsilon^2 C(k)Q_0(k)C^*(k)] dk + O(\epsilon^4). \quad (28)$$

*Remark 9:* Let us stress again the important advantage of the above perturbation analysis in comparison to the method of Theorem 2. In the direct method of computing the  $\mathcal{H}^2$ -norm proposed in Theorem 2, one has to solve a family of *infinite-dimensional* (operator) algebraic Lyapunov equations, whereas the perturbation method reduces the  $\mathcal{H}^2$ -norm computation to that of solving a family of *finite-dimensional* (matrix) Lyapunov/Sylvester equations (24)-(26). ■

In summary, we have the following.

*Main Result of Perturbation Analysis of  $\mathcal{H}^2$ -Norm:* Consider the exponentially stable spatially periodic LTI system  $G$ , with spatial period  $X = 2\pi/\Omega$  and state-space realization (12). Then for small values of  $|\epsilon|$  the  $\mathcal{H}^2$ -norm of the system (12) can be computed from (28), where  $P_0(\cdot)$  and  $Q_0(\cdot)$  are solutions of the family of finite-dimensional Lyapunov/Sylvester equations described by (24)-(26). ■

## VI. EXAMPLES

As an application of the perturbation results of the previous section, we first investigate the occurrence of ‘parametric resonance’ for a class of spatially periodic systems. Parametric resonance occurs when a specific frequency  $\Omega_{\text{res}}$  of the periodic perturbation resonates with some ‘natural frequency’  $\varkappa$  of the unperturbed system, leading to a local (in  $\Omega$ ) change in system behavior [3]. In the systems we consider in this section, this change in behavior is captured by the value of the  $\mathcal{H}^2$ -norm.

*Example 4:* Let us consider the periodic PDE

$$\begin{aligned} \partial_t \psi &= -(\partial_x^2 + \varkappa^2)^2 \psi - c\psi + \epsilon \cos(\Omega x) \psi + u \\ y &= \psi, \end{aligned} \quad (29)$$

with  $0 \neq \varkappa \in \mathbb{R}$  and  $c > 0$ . Comparing (29) and (12) we have

$$A_0(k) = -(k^2 - \varkappa^2)^2 - c, \quad B^o(k) = 1, \quad C^o(k) = 1, \quad B(k) = 1, \quad C(k) = 1, \quad L = \frac{1}{2}.$$

For this system, the functions  $P_0(k)$  and  $Q_0(k)$  of the previous section simplify to<sup>3</sup>

$$P_0(k) = \frac{-1}{2A_0(k)}, \quad (30)$$

$$\begin{aligned} Q_0(k) &= \frac{1}{(A_0(k))^2} \left( \frac{-1}{2A_0(k - \Omega)} + \frac{-1}{2A_0(k + \Omega)} \right) \\ &= 4(P_0(k))^2 (P_0(k - \Omega) + P_0(k + \Omega)), \end{aligned} \quad (31)$$

and it is our aim to find the  $\mathcal{H}^2$ -norm

$$\|G\|_{\mathcal{H}^2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} (P_0(k) + \epsilon^2 Q_0(k)) dk + O(\epsilon^4), \quad (32)$$

for different values of the parameter  $\Omega > 0$ . More specifically, we are interested in the values of  $\Omega$  for which the  $\mathcal{H}^2$ -norm is maximized.

From (30),  $P_0(k) = \frac{1}{2(k^2 - \varkappa^2)^2 + c}$ . The first plot of Figure 4 shows  $P_0(\cdot)$ , while the second shows  $P_0(\cdot - \Omega)$  and  $P_0(\cdot + \Omega)$  (dashed), for a given value of  $\Omega \neq 0$ . As  $\Omega$  is increased,  $P_0(\cdot - \Omega)$  slides to the right and  $P_0(\cdot + \Omega)$  to the left. From (31) it is clear that to find  $Q_0(\cdot)$  for any  $\Omega$ , one would sum the two functions in the second plot and multiply the result by the square of the first plot. The interesting question now is, for what value(s) of  $\Omega \in (0, \infty)$  would the  $\mathcal{H}^2$ -norm in (32) be maximized.

<sup>3</sup>To find  $Q_0(k)$  one needs to first find  $P_1(k)$ , but we have omitted the details for brevity.

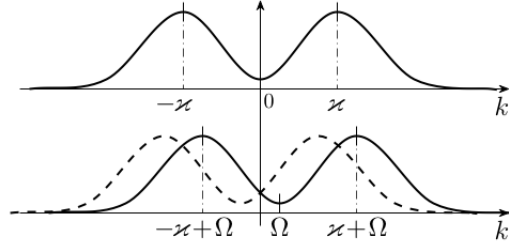


Fig. 4. Above: Plot of  $P_0(\cdot)$ . Below: Plot of  $P_0(\cdot - \Omega)$  and  $P_0(\cdot + \Omega)$  (dashed).

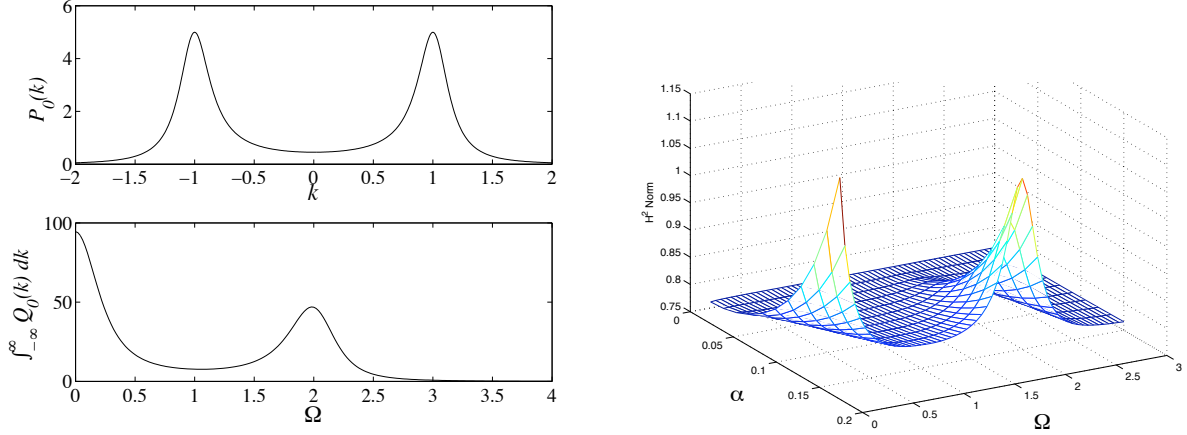


Fig. 5. Left: Plots of Example 4 for  $\varkappa = 1$  and  $c = 0.1$ . Notice that the first graph is plotted against  $k$  and the second against  $\Omega$ . Right: The plot of  $\mathcal{H}^2$ -norm of the same example, but calculated by taking large truncations of the  $\mathcal{A}_\theta$ ,  $\mathcal{B}_\theta$ ,  $\mathcal{C}_\theta$  matrices and using Theorem 2.

One can easily see that as  $\Omega \rightarrow 0$ , the peaks of  $P_0(\cdot - \Omega)$  and  $P_0(\cdot + \Omega)$  merge toward those of  $(P_0(\cdot))^2$ , thus  $\int_{-\infty}^{\infty} Q_0(k) dk$  grows and hence  $\|G\|_{\mathcal{H}^2}^2$  grows.<sup>4</sup> This is not surprising; as  $\Omega \rightarrow 0$ , the perturbation is tending toward a constant function  $F(x) = \cos(\Omega x) \rightarrow 1$ . This results in shifting the whole spectrum of  $A^\circ$  toward the right-half of the complex plane, thus increasing the  $\mathcal{H}^2$ -norm of the perturbed system.

But we are more interested in frequencies  $\Omega \gg 0$  that lead to a local (in  $\Omega$ ) increase in the  $\mathcal{H}^2$ -norm. Now notice that a different alignment of the peaks can also occur, which leads to another local maximum of the  $\mathcal{H}^2$ -norm as a function of  $\Omega$ . This happens when the peak of  $P_0(\cdot - \Omega)$  at  $k = -\varkappa + \Omega$  becomes aligned with the peak of  $(P_0(\cdot))^2$  at  $k = \varkappa$ , and, simultaneously, the peak of  $P_0(\cdot + \Omega)$  at  $k = \varkappa - \Omega$  becomes aligned with the peak of  $(P_0(\cdot))^2$  at  $k = -\varkappa$ . Clearly this occurs when

$$-\varkappa + \Omega_{\text{res}} = \varkappa \implies \Omega_{\text{res}} = 2\varkappa.$$

This result agrees exactly with that obtained in [12], where in the analysis of the same problem it is shown that the part of the spectrum of  $A$  closest to the imaginary axis ‘resonates’ with perturbations whose frequency satisfies the relation  $\Omega = 2\varkappa$ .

Consider (29) with  $\varkappa = 1$  and  $c = 0.1$ . For this system  $\int_{-\infty}^{\infty} P_0(k) dk \approx 4.74$ . Figure 5 (left) shows the graphs of  $P_0(\cdot)$  (plotted against  $k$ ) and  $\int_{-\infty}^{\infty} Q_0(k) dk$  (plotted against  $\Omega$ ). The peak at  $\Omega = 2$  in the lower plot verifies the result  $\Omega_{\text{res}} = 2\varkappa$  obtained above.

Figure 5 (right) shows the  $\mathcal{H}^2$ -norm of this system computed by taking large enough truncations of the  $\mathcal{A}_\theta$ ,  $\mathcal{B}_\theta$ ,  $\mathcal{C}_\theta$  matrices and then applying Theorem 2. The figure shows that the trends were indeed correctly predicted by the perturbation analysis; the peaks at  $\Omega = 0, 2$  correspond to those of  $\int_{-\infty}^{\infty} Q_0(k) dk$ .

<sup>4</sup>Remember that  $P_0(k)$  is independent of  $\Omega$ , and thus  $\int_{-\infty}^{\infty} P_0(k) dk$  remains constant for different  $\Omega$ .

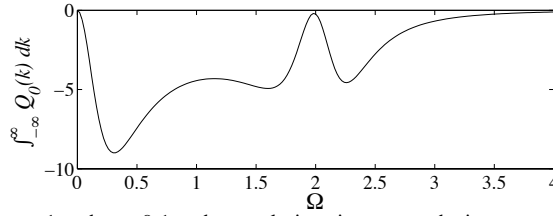


Fig. 6. The plot of Example 4 with  $\varkappa = 1$  and  $c = 0.1$  and a purely imaginary perturbation.

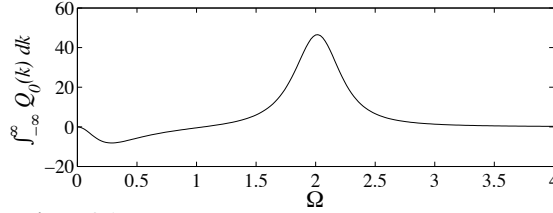


Fig. 7. Plot of Example 5 for  $\varkappa = 1$  and  $c = 0.1$ .

Now consider (29) with  $\varkappa = 1$  and  $c = 0.1$ , but with  $\epsilon$  replaced by  $\epsilon j$  (i.e., a purely imaginary perturbation). Obviously the unperturbed system remains the same as before and hence  $\int_{-\infty}^{\infty} P_0(k) dk \approx 4.74$ . Figure 5 shows the graph of  $\int_{-\infty}^{\infty} Q_0(k) dk$ , which demonstrates that for this system the purely imaginary perturbation reduces the  $\mathcal{H}^2$ -norm at all frequencies. We address the physical interpretation of such a perturbation in the Appendix. ■

We continue with more examples to demonstrate that by appropriately choosing the frequency of the perturbation, one can *decrease* or (as in the previous example) increase the  $\mathcal{H}^2$ -norm of the unperturbed system.

*Example 5:* Let us consider the periodic PDE [5]

$$\begin{aligned} \partial_t \psi &= -(\partial_x^2 + \varkappa^2)^2 \psi - c\psi + \epsilon \cos(\Omega x) \partial_x \psi + u \\ y &= \psi. \end{aligned} \quad (33)$$

Comparing (33) and (12) we have

$$A_0(k) = -(k^2 - \varkappa^2)^2 - c, \quad B^0(k) = 1, \quad C^0(k) = jk, \quad B(k) = 1, \quad C(k) = 1, \quad L = \frac{1}{2}.$$

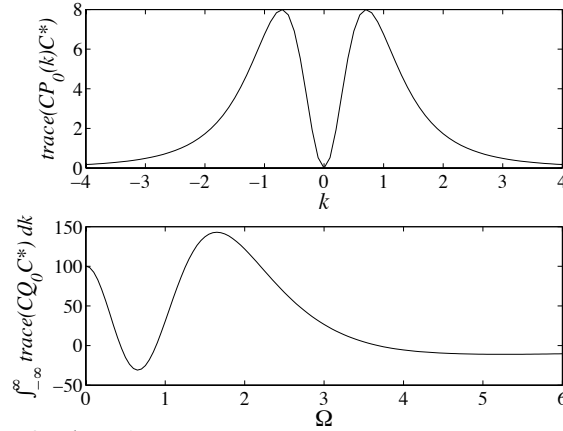
The difference between this system and (29) is that here  $C^0$  is a spatial derivative. The plot of Figure 7 demonstrates  $\int_{-\infty}^{\infty} Q_0(k) dk$  for  $\varkappa = 1$  and  $c = 0.1$ . Notice that the peak at  $\Omega_{\text{res}} = 2$  remains the same as in Figure 5, but we now have a decrease at small frequencies. This is due to the derivative operator  $C^0 = \partial_x$ . Finally, notice the agreement of the perturbation analysis here with the (non-perturbation) calculations for the same system in the Third Example of Section VII in [5]. Our perturbation methods correctly predict the increase at  $\Omega = 2$  and the decrease around  $\Omega \approx 0.4$  of the  $\mathcal{H}^2$ -norm. ■

*Example 6:* The following system is motivated by channel flow problems

$$A_0(k) = \begin{bmatrix} -\frac{1}{R}(k^2 + c) & 0 \\ jk & -\frac{1}{R}(k^2 + c) \end{bmatrix}$$

$$B^0(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C^0(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(k) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C(k) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We perform numerical calculations for  $R = 6$ ,  $c = 1$ . We have  $\int_{-\infty}^{\infty} \text{trace}[C(k)P_0(k)C^*(k)] dk \approx 20.72$ . Figure 8 shows that the  $\mathcal{H}^2$ -norm can be decreased by the application of periodic perturbations with frequency  $\Omega \approx 0.7$ . It is interesting that if one uses the locations of the peaks in the first plot of Figure 8 to find  $\varkappa = 0.75$ , then from the peak at  $\Omega_{\text{res}} = 1.6$  in the second plot it seems that the relationship  $\Omega_{\text{res}} \approx 2\varkappa = 1.5$  still holds with an acceptable error even for this *non-scalar* example. ■

Fig. 8. Graphs of Example 6 for  $R = 6$  and  $c = 1$ .

## VII. STABILITY AND THE SPECTRUM-DETERMINED GROWTH CONDITION

In the literature on semigroups, there exist examples in which  $\Sigma(A)$  lies entirely inside  $\mathbb{C}^-$ , but  $\|e^{At}\|$  does not decay exponentially; see [6] and more recently [7]. In such cases it is said that the semigroup does not satisfy the *spectrum-determined growth condition* [8]. The determining factor in the examples presented in [6] and [7] can be interpreted as the accumulation of the eigenvalues of  $A_\theta$  around  $\pm j\infty$  in the form of Jordan blocks of ever-increasing size (i.e. as the eigenvalues tend to  $\pm j\infty$  their algebraic multiplicity increases while their geometric multiplicity stays equal to one). But such cases are ruled out when one deals with holomorphic semigroups, which is the reason we consider these semigroups in this paper.

Our ultimate aim in this section is to verify exponential stability. By Theorem 1, in order to prove exponential stability of a holomorphic semigroup with infinitesimal generator  $A$ , it is sufficient to show that  $\Sigma(A) \subset \mathbb{C}^-$ . Hence, in the first part of this section, we give conditions under which the  $A$  operators described by (12) generate holomorphic semigroups. In the second part, we find sufficient conditions which guarantee  $\Sigma(A) \subset \mathbb{C}^-$ .

Once again, the setup is that of (12) where  $\epsilon$  is a small complex scalar. In addition, assume that  $A_0(k) \in \mathbb{C}^{q \times q}$  is diagonalizable for every  $k \in \mathbb{R}$ .

### Conditions for Sectorial Infinitesimal Generator

To find conditions under which  $A$  in (12) will define a holomorphic semigroup we have to check the condition (18). Since the Fourier transform preserves norms, (18) is equivalent to checking  $\|(zI - \hat{A})^{-1}\| \leq M/|z - \alpha|$  for  $z$  belonging to some sector of  $\mathbb{C}$ . This involves finding the inverse of the operator  $zI - \hat{A}$  and then calculating its norm. Such a computation can be very difficult since  $(zI - \hat{A})^{-1}$  has the form (5) [also depicted in Figure 1 (right)], i.e., it has an infinite number of impulse sheets. On the other hand, finding  $\|(zI - \hat{A}^0)^{-1}\|$  is very easy, due to the diagonal structure of  $\hat{A}^0$ . Indeed  $\|(zI - \hat{A}^0)^{-1}\| = \sup_{k \in \mathbb{R}} \|(zI - A_0(k))^{-1}\|$ .

Thus to establish conditions for  $A$  to be sectorial, we again use perturbation theory. We first find conditions under which  $A^0$  is sectorial. We then show that  $A = A^0 + \epsilon E$  remains sectorial if  $E$  is ‘weaker’ than  $A^0$  in a certain sense we will describe, and if  $\epsilon$  is small enough.

In the next theorem we present a condition for a spatially invariant  $A^0$  with symbol  $A_0(k)$  to be sectorial.

**Theorem 3:** Let  $A_0(k)$  be diagonalizable for every  $k \in \mathbb{R}$ , and let  $U(k)$  be the transformation that diagonalizes  $A_0(k)$ , i.e.,  $A_0(k) = U(k) \Lambda(k) U^{-1}(k)$  with  $\Lambda(k)$  diagonal. Let  $\kappa(k) := \|U(k)\| \|U^{-1}(k)\|$  denote the condition number of  $U(k)$ . If  $\sup_{k \in \mathbb{R}} \kappa(k) < \infty$ , and for every  $k \in \mathbb{R}$  the resolvent set  $\rho(A_0(k))$  contains a sector of the complex plane  $|\arg(z - \alpha)| < \frac{\pi}{2} + \gamma$  with  $\gamma > 0$  and  $\alpha \in \mathbb{R}$  both independent of  $k$ , then  $A^0$  is sectorial.

*Proof:* See Appendix. ■

This theorem has a particularly simple interpretation when  $A_0(\cdot)$  is scalar. In this case  $\kappa(U(k)) = 1$  for all  $k \in \mathbb{R}$ . Now since  $A_0(\cdot)$  traces a curve in the complex plane, by Theorem (3) if this curve stays outside some sector  $|\arg(z - \alpha)| \leq \frac{\pi}{2} + \gamma$ ,  $\gamma > 0$ , of the complex plane then  $A^0$  is sectorial.

The following theorem is from [11]. It uses the notion of *relative boundedness* [9] of one unbounded operator with respect to another unbounded operator.

*Theorem 4:* Suppose  $A^\circ \in \mathcal{H}(\gamma, \alpha, M)$  and  $E = B^\circ F C^\circ$  is relatively bounded with respect to  $A^\circ$  so that

$$\|E\psi\| \leq a\|\psi\| + b\|A^\circ\psi\|, \quad \psi \in \mathcal{D}, \quad (34)$$

with  $0 \leq a < \infty$  and  $0 \leq b|\epsilon| < 1/(1 + M)$ . Then  $A = A^\circ + \epsilon E$  is a sectorial operator. ■

This theorem says that if  $A^\circ$  is sectorial, then so is  $A = A^\circ + \epsilon E$  if  $E$  is weaker than  $A^\circ$  in the sense of (34) and if  $|\epsilon|$  is small enough. Notice that at this point, condition (34) can not be reduced to a condition in terms of Fourier symbols (i.e. a condition that can be checked pointwise in  $k$ ) as in Theorem 3. This is because  $E$  is not spatially invariant. But once the exact form of the operators  $B^\circ$  and  $C^\circ$  is known, (34) can be simplified to a condition on the symbols of  $A^\circ$ ,  $B^\circ$  and  $C^\circ$ . Let us clarify this statement with the aid of an example.

*Example 7:* Consider the periodic PDE

$$\begin{aligned} \partial_t \psi &= -(\partial_x^2 + \varkappa^2)^2 \psi - c\psi + \epsilon \partial_x \cos(\Omega x) \psi + u \\ y &= \psi. \end{aligned}$$

It is easy to see that  $A^\circ = -(\partial_x^2 + \varkappa^2)^2 - c$ ,  $B^\circ = \partial_x$  and  $C^\circ = \delta(x)$  (the identity convolution operator). By formal differentiation we have

$$E\psi = \partial_x \cos(\Omega x) \psi = -\Omega \sin(\Omega x) \psi + \cos(\Omega x) \partial_x \psi.$$

Using the triangle inequality and  $\|\sin(\Omega x)\| = \|\cos(\Omega x)\| = 1$  we have

$$\|E\psi\| \leq |\Omega| \|\psi\| + \|\partial_x \psi\|. \quad (35)$$

Thus we have effectively ‘commuted out’ the (bounded) spatially periodic operator in  $E$ , and are left with only spatially invariant operators on the right of (35). Hence, after applying a Fourier transformation to the right side of (35), a sufficient condition for (34) to hold is that

$$|\Omega| + |k| \leq a + b|(k^2 - \varkappa^2)^2 + c|, \quad k \in \mathbb{R},$$

which can be shown to be satisfied for large enough  $a > |\Omega|$  and  $b > 0$ . ■

*Remark 10:* The above example makes clear the notion of  $E$  being ‘weaker’ than  $A^\circ$  that we mentioned at the beginning of this subsection. If in Example 7 we had  $B^\circ = \partial_x^\nu$  and  $C^\circ = \partial_x^\mu$  and  $\nu + \mu = 5$ , then  $E$  would contain a 5<sup>th</sup> order derivative, whereas the highest order of  $\partial_x$  in  $A^\circ$  is 4. This would mean that (34) can not be satisfied for any choice of  $a$  and  $b$ . ■

#### Conditions for Infinitesimal Generator to have Spectrum in $\mathbb{C}^-$

The final step in establishing exponential stability is to show that  $\Sigma(A) \subset \mathbb{C}^-$ . Unfortunately it is in general very difficult to find the spectrum of an infinite-dimensional operator. Thus we proceed as follows. We consider the (block) diagonal operators  $\mathcal{A}_\theta^\circ$ ,  $\theta \in [0, \Omega)$ . This allows us to extend Geršgorin-type methods (existing for finite-dimensional matrices) to find bounds on the location of  $\Sigma(\mathcal{A}_\theta)$ ,  $\mathcal{A}_\theta = \mathcal{A}_\theta^\circ + \epsilon \mathcal{E}_\theta$ . In turn, we use this to find conditions under which  $\Sigma(\mathcal{A}_\theta) \subset \mathbb{C}^-$ , and thus  $\Sigma(A) \subset \mathbb{C}^-$ .

In locating the spectrum of a finite-dimensional matrix  $T \in \mathbb{C}^{q \times q}$ , the theory of Geršgorin circles [18] provides us with a region of the complex plane that contains the eigenvalues of  $T$ . This region is composed of the union of  $q$  disks, the centers of which are the diagonal elements of  $T$ , and their radii depend on the magnitude of the off-diagonal elements [18]. This theory has also been extended to the case of finite-dimensional block matrices (i.e., matrices whose elements are themselves matrices) in [19]. Next, we further extend this theory to include bi-infinite (block) matrices  $\mathcal{A}_\theta$ .

For the operator  $A = A^\circ + \epsilon E$ , take  $\mathfrak{B}_k$  to be the set of complex numbers  $z$  that satisfy

$$\sigma_{\min}(zI - A_0(k)) \leq |\epsilon| (\|A_{-1}(k)\| + \|A_1(k)\|), \quad (36)$$

where  $\sigma_{\min}(zI - A_0(k))$  denotes the smallest singular value of the matrix  $zI - A_0(k)$ .



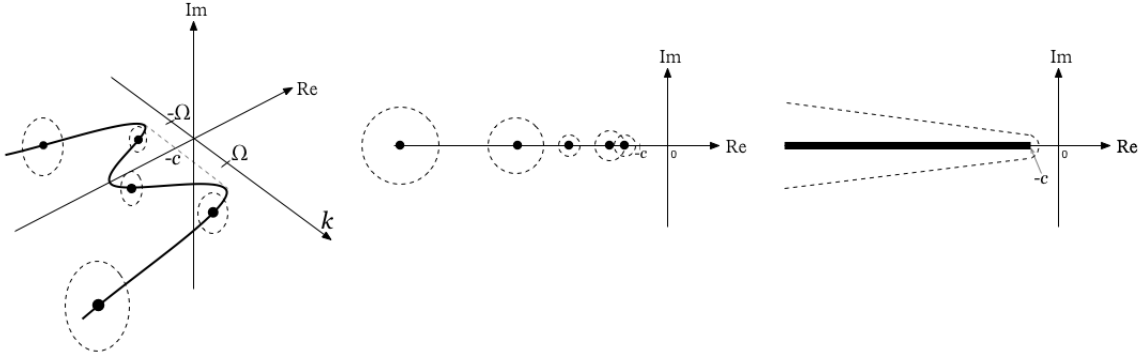


Fig. 9. Left: The  $\mathfrak{B}_{\theta_n}$  regions viewed in the ‘complex-plane  $\times$  spatial-frequency’ space (the disks are parallel to the complex plane). Center:  $\Sigma(\mathcal{A}_\theta)$  is contained inside the union of the regions  $\mathfrak{B}_{\theta_n}$ . Right: The bold line shows  $\Sigma(A^\circ)$  and the dotted region contains  $\Sigma(A)$ ,  $A = A^\circ + \epsilon E$ .

*Lemma 5:* For every  $\theta$  we have  $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta$ , where

$$\mathfrak{S}_\theta = \overline{\bigcup_{n \in \mathbb{Z}} \mathfrak{B}_{\theta_n}}.$$

*Proof:* See Appendix. ■

*Example 8:* Let us consider the periodic PDE [5]

$$\begin{aligned} \partial_t \psi &= -(\partial_x^2 + \varkappa^2)^2 \psi - c \psi + \epsilon \cos(\Omega x) \partial_x \psi + u \\ y &= \psi. \end{aligned} \tag{37}$$

We have

$$A_0(k) = -(k^2 - \varkappa^2)^2 - c, \quad B^0(k) = 1, \quad C^0(k) = jk, \quad B(k) = 1, \quad C(k) = 1, \quad L = \frac{1}{2}.$$

From (15),  $A_1(k) = \frac{j}{2}(k - \Omega)$ ,  $A_{-1}(k) = \frac{j}{2}(k + \Omega)$ , and thus  $\|A_{-1}(k)\| + \|A_1(k)\| = \frac{1}{2}(|k - \Omega| + |k + \Omega|)$ . Hence (36) leads to

$$\sigma_{\min}(zI - A_0(k)) = |zI - A_0(k)| \leq \frac{|\epsilon|}{2} (|k - \Omega| + |k + \Omega|) = \begin{cases} \Omega |\epsilon| & |k| \leq \Omega \\ |k| |\epsilon| & |k| \geq \Omega \end{cases},$$

which means that the set  $\mathfrak{S}_\theta$  is composed of the union of disks with centers at  $A_0(\theta_n)$  and radii  $\frac{|\epsilon|}{2}(|\theta_n - \Omega| + |\theta_n + \Omega|)$ . This is nothing but an extension of the classical Geršgorin disks to bi-infinite matrices. Figure 9 (left & center) show  $\mathfrak{S}_\theta$  in the complex-plane  $\times$  spatial-frequency space and in  $\mathbb{C}$  respectively.<sup>5</sup> ■

*Remark 11:* The set

$$\begin{aligned} \Sigma_\epsilon(M) &:= \{z \in \mathbb{C} \mid \sigma_{\min}(zI - M) \leq \epsilon\} \\ &\equiv \{z \in \mathbb{C} \mid \|(zI - M)\varphi\| \leq \epsilon \text{ for some } \|\varphi\| = 1\} \\ &\equiv \{z \in \mathbb{C} \mid z \in \Sigma_p(M + Z) \text{ for some } \|Z\| \leq \epsilon\} \end{aligned} \tag{38}$$

is called the  $\epsilon$ -pseudospectrum of the matrix  $M$  [20]. Clearly  $\Sigma_{\epsilon'}(M) \subseteq \Sigma_\epsilon(M)$  if  $\epsilon' \leq \epsilon$ , and  $\Sigma_\epsilon(M) = \Sigma_p(M)$  for  $\epsilon = 0$ . The pseudospectrum is composed of small sets around the eigenvalues of  $M$ . For instance if  $M$  has simple eigenvalues, then for small enough values of  $\epsilon$  the pseudospectrum consists of disjoint compact and convex neighborhoods of each eigenvalue [21]. ■

<sup>5</sup>We would like to point out that Figure 9 (left) is technically incorrect; once the spatially invariant system is perturbed by a spatially periodic perturbation it is no longer spatially invariant and thus can not be represented by a Fourier symbol. Hence its spectrum can no longer be demonstrated in the complex-plane  $\times$  spatial-frequency space. Figure 9 (center) demonstrates the correct representation of the Geršgorin disks for  $\mathcal{A}_\theta$ .

*Remark 12:* Comparing (38) and the definition of  $\mathfrak{B}_k$  in (36), it is clear that  $\mathfrak{B}_k = \Sigma_\varepsilon(A_0(k))$  with  $\varepsilon = |\epsilon| (\|A_{-1}(k)\| + \|A_1(k)\|)$ . Thus for every  $k \in \mathbb{R}$ , (36) defines a closed region of  $\mathbb{C}$  that includes the eigenvalues of  $A_0(k)$ . ■

We now employ Lemma 5 to determine whether  $\Sigma(A)$  resides completely inside  $\mathbb{C}^-$ , as needed to assess system stability. Take  $\mathfrak{D}_\varepsilon$  to be the closed disk of radius  $\varepsilon$  and center at the origin, and  $\mathfrak{B}_k$  to be the region described by (36). Define the sum of sets by  $\mathfrak{U}_1 + \mathfrak{U}_2 = \{z \mid z = z_1 + z_2, z_1 \in \mathfrak{U}_1, z_2 \in \mathfrak{U}_2\}$ . Also, for every  $k \in \mathbb{R}$  let  $\lambda_{\max}(k)$  represent the eigenvalue of  $A_0(k)$  with the maximum real part, and let  $\kappa(k)$  be defined as in Theorem 3.

*Theorem 6:* For every  $k$ ,  $\mathfrak{B}_k$  is contained inside  $\Sigma_p(A_0(k)) + \mathfrak{D}_{r(k)}$  with

$$r(k) = |\epsilon| (\|A_{-1}(k)\| + \|A_1(k)\|) \kappa(k).$$

In particular, if  $\Sigma(A^0) \subset \mathbb{C}^-$  and

$$r(k) < |\operatorname{Re}(\lambda_{\max}(k))| + \beta \quad (39)$$

for every  $k \in \mathbb{R}$  and some  $\beta < 0$  independent of  $k$ , then  $\Sigma(A) \subset \mathbb{C}^-$ .

*Proof:* See Appendix. ■

*Example 9:* Once again we use the scalar system of Example 8.  $\kappa(k) = 1$  since  $A_0(k)$  is scalar,  $|\operatorname{Re}(\lambda_{\max}(k))| = |(k^2 - \varkappa^2)^2 + c|$ , and

$$\|A_{-1}(k)\| + \|A_1(k)\| = \frac{1}{2} (|k - \Omega| + |k + \Omega|).$$

Thus condition (39) becomes

$$\frac{|\epsilon|}{2} (|k - \Omega| + |k + \Omega|) < |(k^2 - \varkappa^2)^2 + c| + \beta.$$

If this condition is satisfied for some  $\beta < 0$ , the dotted region in Figure 9 (right) will remain inside  $\mathbb{C}^-$  and thus  $\Sigma(A) \subset \mathbb{C}^-$ . ■

To recap, to assess exponential stability we first find sufficient conditions on  $A$  such that it belongs to the class of operators for which the spectrum-determined growth condition holds. These are conditions under which  $A$  is sectorial. We then find sufficient conditions for  $A$  to have  $\mathbb{C}^-$  spectrum. We do this via an extension of Geršgorin circles to bi-infinite (block) matrices.

### VIII. CONCLUSIONS AND FUTURE WORK

We use perturbation analysis to find a computationally efficient way of revealing trends in the  $\mathcal{H}^2$ -norm of spatially periodic systems. We show that for certain classes of systems, the periodicity can be chosen so as to increase the  $\mathcal{H}^2$ -norm or induce parametric resonance. An application of this would be in mixing problems. It is also shown that the  $\mathcal{H}^2$ -norm can be made to decrease for an appropriate choice of the frequency of the perturbation. This would be the desired scenario in the design of the body of aircraft. We demonstrate that for certain scalar systems, the value of the spatial period that achieves the desired increase/decrease of the  $\mathcal{H}^2$ -norm can be characterized exactly based on the description of the nominal system.

The methods presented here can also be used in systems with many spatial directions. For example, consider the PDE

$$\psi_t = \psi_{yy} + \psi_{xx} + c\psi + \epsilon \cos(\Omega x)\psi$$

with  $y \in [-1, 1]$  and  $x \in \mathbb{R}$ . To put this system into the developed framework one would only have to perform a discrete approximation of the operator  $\partial_y^2$  with the appropriate boundary conditions.

We also study the problem of checking the exponential stability of a spatially periodic system. We do this by (i) finding conditions under which the  $A$ -operator is sectorial (i.e., generates a holomorphic semigroup) and thus satisfies the spectrum-determined growth condition, and (ii) deriving conditions which guarantee that  $A$  has spectrum contained inside the open left-half of the complex plane.

Future research in this direction would include an exact (analytical) characterization of the frequencies for which the  $\mathcal{H}^2$ -norm is most increased/decreased for the general case of matrix-valued  $\hat{A}(\cdot)$ . The perturbation methods presented here could also be generalized to bi-infinite Sylvester equations which arise frequently in fluids problems.

## IX. APPENDIX

*Proof of Theorem 3*

It is shown in [22] that a sufficient condition for  $A^\circ$  to be sectorial is that  $\rho(A^\circ)$  contain some right half plane  $\{z \in \mathbb{C} \mid \operatorname{Re}(z) \geq \mu\}$ , and

$$\|z(zI - A^\circ)^{-1}\| \leq M \quad \text{for} \quad \operatorname{Re}(z) \geq \mu,$$

for some  $\mu \geq 0$  and  $M \geq 1$ .

Now since  $A_0(k) \in \mathbb{C}^{q \times q}$  has simple eigenvalues for every  $k$ , there exists a matrix  $U(k)$  such that  $A_0(k) = U(k) \Lambda(k) U^{-1}(k)$  with  $\Lambda(k)$  a diagonal matrix. Let  $\lambda_i(k)$ ,  $i = 1, \dots, q$  denote the diagonal elements of  $\Lambda(k)$ . Clearly these are also the eigenvalues of  $A_0(k)$ . Thus we have

$$\begin{aligned} \|z(zI - A^\circ)^{-1}\| &\leq \sup_{k \in \mathbb{R}} \left( \|z(zI - A_0(k))^{-1}\| \right) \\ &\leq \sup_{k \in \mathbb{R}} \left( \|U(k)\| \|U^{-1}(k)\| \|z(zI - \Lambda(k))^{-1}\| \right) \\ &= \sup_{k \in \mathbb{R}} \left( \kappa(k) \frac{|z|}{\operatorname{dist}[z, \Sigma_p(A_0(k))]} \right) \\ &\leq \kappa_{\max} \sup_{k \in \mathbb{R}} \left( \frac{|z|}{\operatorname{dist}[z, \Sigma_p(A_0(k))]} \right), \end{aligned}$$

where  $\kappa_{\max} := \sup_{k \in \mathbb{R}} \kappa(k)$ .

Let us now choose the positive scalar  $M' = (1 + \kappa_{\max})M$ ,  $M > 1$ , and consider for a given  $k$  the region of the complex plane where

$$\kappa_{\max} \frac{|z|}{\operatorname{dist}[z, \Sigma_p(A_0(k))]} \geq M'.$$

This region (which contains the eigenvalues  $\lambda_i(k)$ ) is contained inside the union of the circles

$$\kappa_{\max} \frac{|z|}{|z - \lambda_i(k)|} \geq M', \quad i = 1, \dots, q,$$

which are themselves contained inside the larger circles

$$|z - \lambda_i(k)| \leq \frac{|\lambda_i(k)|}{M}, \quad i = 1, \dots, q. \quad (\text{A1})$$

Notice that (A1) describes circles whose radii increase like  $|\lambda_i(k)|/M$ ,  $M > 1$ , as their centers  $\lambda_i(k)$  become distant from the origin. Clearly a sufficient condition for these circles to belong to some open half plane  $\{z \in \mathbb{C} \mid \operatorname{Re}(z) < \mu\}$  for all  $k \in \mathbb{R}$  and large enough  $M$  is that  $\Sigma_p(A_0(k))$ ,  $k \in \mathbb{R}$ , reside outside some sector  $|\arg(z - \alpha)| \leq \frac{\pi}{2} + \gamma$ ,  $\gamma > 0$ , of the complex plane.

Finally, if the circles (A1) are contained in some open half plane  $\{z \in \mathbb{C} \mid \operatorname{Re}(z) < \mu\}$  for all  $k \in \mathbb{R}$ , then for  $\operatorname{Re}(z) \geq \mu$ ,  $z \in \rho(A_0(k))$  and we have

$$\kappa_{\max} \sup_{k \in \mathbb{R}} \left( \frac{|z|}{\operatorname{dist}[z, \Sigma_p(A_0(k))]} \right) \leq M$$

and thus  $\|z(zI - A^\circ)^{-1}\| \leq M$  for  $\operatorname{Re}(z) \geq \mu$ .

*Proof of Lemma 5*

We use  $\Pi_N T \Pi_N$  to denote the  $(2N + 1) \times (2N + 1)$  [block] truncation of an operator  $T$  on  $\ell^2$ , where  $\Pi_N$  is the projection defined by

$$\operatorname{diag} \left\{ \dots, 0, \underbrace{I, \dots, I}_{\substack{\text{center} \\ \downarrow \\ 2N+1 \text{ times}}}, 0, \dots \right\},$$

where  $I$  is the identity matrix. Notice that  $\Pi_N T \Pi_N$  is still an operator on  $\ell^2$ ; it made from the bi-infinite  $T$  by replacing all entries outside the center  $(2N+1) \times (2N+1)$  block with zeros. We now form the finite-dimensional matrix  $\Pi_N \mathcal{A}_\theta \Pi_N|_{\Pi_N \ell^2}$  by restricting  $\Pi_N \mathcal{A}_\theta \Pi_N$  to the finite-dimensional space  $\Pi_N \ell^2$ . Clearly  $\Pi_N \mathcal{A}_\theta \Pi_N|_{\Pi_N \ell^2}$  has pure point spectrum. Hence using a generalized form of the Geršgorin Circle Theorem [19] for finite-dimensional (block) matrices, we conclude that

$$\Sigma(\Pi_N \mathcal{A}_\theta \Pi_N|_{\Pi_N \ell^2}) \subset \bigcup_{|n| \leq N} \mathfrak{B}_{\theta_n} \subseteq \overline{\bigcup_{n \in \mathbb{Z}} \mathfrak{B}_{\theta_n}}$$

where  $\mathfrak{B}_{\theta_n}$  are regions of  $\mathbb{C}$  defined by (36). Since this holds for all  $N \geq 0$ , we have  $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta$ .

#### Proof of Theorem 6

If  $U(k)$  diagonalizes  $A_0(k)$ ,  $A_0(k) = U(k) \Lambda(k) U^{-1}(k)$ , and  $\kappa(k) = \|U(k)\| \|U^{-1}(k)\|$  denotes the condition number of  $U(k)$ , then from [23] the pseudospectrum of  $A_0(k)$  satisfies

$$\Sigma_p(A_0(k)) + \mathfrak{D}_\varepsilon \subseteq \Sigma_\varepsilon(A_0(k)) \subseteq \Sigma_p(A_0(k)) + \mathfrak{D}_{\varepsilon \kappa(k)} \quad (\text{A2})$$

for all  $\varepsilon \geq 0$ . Thus the first statement of the Theorem follows immediately from (A2) and the fact that  $\mathfrak{B}_k = \Sigma_\varepsilon(A_0(k))$  with  $\varepsilon = |\epsilon| (\|A_{-1}(k)\| + \|A_1(k)\|)$  [see Remark 12]. To prove the second statement, let  $\mathbb{C}_\beta^-$  denote all complex numbers with real part less than  $\beta \in \mathbb{R}$ . It follows from  $\Sigma(A^0) \subset \mathbb{C}^-$  that  $\Sigma(\mathcal{A}_\theta^0) \subset \mathbb{C}^-$  for every  $\theta$ . If (39) holds then

$$\mathfrak{B}_{\theta_n} \subseteq \Sigma_p(A_0(\theta_n)) + \mathfrak{D}_{r(\theta_n)} \subset \mathbb{C}_\beta^-$$

for every  $n \in \mathbb{Z}$ , and from Lemma 5 we have  $\Sigma(\mathcal{A}_\theta) \subseteq \mathfrak{S}_\theta = \overline{\bigcup_{n \in \mathbb{Z}} \mathfrak{B}_{\theta_n}} \subset \mathbb{C}_{\beta'}^-$  for some  $\beta < \beta' < 0$  and every  $\theta$ . Thus  $\Sigma(A) \subset \mathbb{C}^-$ .

#### Interpretation of Imaginary States

It was shown above that one can decrease the  $\mathcal{H}^2$ -norm of certain systems by choosing the perturbation amplitude  $A_1$  to be purely imaginary (or, in general, skew-symmetric). This would yield a perturbed system that can in general have states with nonzero imaginary parts. One could then ask the physical interpretation of such a system.

For any operator  $A_p$  and function  $\psi$  one can write

$$\begin{aligned} A_p &= A_r + jA_i, & A_r, A_i &\in \mathbb{R}^{n \times n}, \\ \psi &= \psi_r + j\psi_i, & \psi_r, \psi_i &\in \mathbb{R}^n. \end{aligned}$$

Then the system equations can be written as

$$\begin{aligned} \partial_t(\psi_r + j\psi_i) &= (A_r + jA_i)(\psi_r + j\psi_i) \\ &\Updownarrow \\ \begin{cases} \partial_t \psi_r &= A_r \psi_r - A_i \psi_i \\ \partial_t \psi_i &= A_r \psi_i + A_i \psi_r \end{cases} \\ &\Updownarrow \\ \partial_t \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} &= \begin{bmatrix} A_r & -A_i \\ A_i & A_r \end{bmatrix} \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix}. \end{aligned}$$

Clearly the state dimension is twice that of the original system with imaginary coefficients, but now  $\begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} \in \mathbb{R}^{2n}$ .

Let us give a simple example. Assume the heat equation, with  $A_1 = j\epsilon$ ,  $\epsilon \in \mathbb{R}$ , i.e.  $A_p = \partial_x^2 - c + j\epsilon \cos(\Omega x)$ . Then  $A_r = A_0 = \partial_x^2 - c$ ,  $A_i = A_1 = j\epsilon \cos(\Omega x)$ , and thus

$$\begin{aligned} \partial_t \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} &= \begin{bmatrix} \partial_x^2 - c & -\epsilon \cos(\Omega x) \\ \epsilon \cos(\Omega x) & \partial_x^2 - c \end{bmatrix} \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} \\ &= \left( \begin{bmatrix} \partial_x^2 - c & 0 \\ 0 & \partial_x^2 - c \end{bmatrix} + \epsilon \cos(\Omega x) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \psi_r \\ \psi_i \end{bmatrix} \end{aligned}$$

which describes two identical systems coupled through the periodic perturbation.

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